

On C^* -algebras related to asymptotic homomorphisms

*

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Abstract

We study the C^* -algebras related to Mishchenko's version of asymptotic homomorphisms. In particular we show that their different versions are weakly homotopy equivalent but not isomorphic to each other. We give also the continuous version for these algebras.

1 Introduction

Let M_k , $k \in \mathbf{N}$, denote the $k \times k$ matrix C^* -algebra. To make notation uniform we will write M_∞ for the C^* -algebra K of compact operators. Let $\kappa = \{\kappa(1), \kappa(2), \dots\}$ be a function from \mathbf{N} to $\mathbf{N} \cup \{\infty\}$, i.e. κ is a sequence with $\kappa(i)$, $i \in \mathbf{N}$, being either integers or infinity. We suppose that the sequences $\kappa = \{\kappa(i)\}$ are *monotonely non-decreasing* and either stabilize (there exists $\lim_{i \rightarrow \infty} \kappa(i)$) or approach infinity as $i \rightarrow \infty$.

Consider the quotient C^* -algebra

$$\prod_{i=1}^{\infty} M_{\kappa(i)} / \oplus_{i=1}^{\infty} M_{\kappa(i)},$$

which is the target C^* -algebra for discrete asymptotic homomorphisms [1, 2], i.e. any discrete asymptotic homomorphism from a C^* -algebra A into matrix algebras or into compacts gives a genuine $*$ -homomorphism from A into this target algebra. In [4] Mishchenko suggested to consider another version of discrete asymptotic homomorphisms, namely to add the following property: if $\varphi_n : A \rightarrow B$ is a discrete asymptotic homomorphism, it should satisfy also

$$\lim_{i \rightarrow \infty} \|\varphi_{i+1}(a) - \varphi_i(a)\| = 0$$

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for any $a \in A$. The corresponding target C^* -algebras were constructed in [3] as follows. Since any sequence κ is non-decreasing, there is a natural inclusion $M_{\kappa(i)} \subset M_{\kappa(i+1)}$, hence the right shift

$$\alpha : (m_1, m_2, \dots) \longmapsto (0, m_1, m_2, \dots), \quad (1)$$

where $m = (m_1, m_2, \dots) \in \prod_{i=1}^{\infty} M_{\nu(i)}$, is well-defined both on $\prod_{i=1}^{\infty} M_{\kappa(i)}$ and on $\prod_{i=1}^{\infty} M_{\kappa(i)} / \oplus_{i=1}^{\infty} M_{\kappa(i)}$. The map α is an endomorphism of these C^* -algebras, and it is easy to see that the α -invariant subset in $\prod_{i=1}^{\infty} M_{\kappa(i)} / \oplus_{i=1}^{\infty} M_{\kappa(i)}$ is a C^* -algebra too. We denote this α -invariant C^* -algebra by $Q(\kappa)$.

If the sequence κ stabilizes, $\lim_{i \rightarrow \infty} \kappa(i) = n$, $n \in \mathbf{N} \cup \{\infty\}$, we denote such κ by \mathbf{n} (or by ∞ if $n = \infty$). In this case we write $Q(\mathbf{n})$ instead of $Q(\kappa)$. The C^* -algebra $Q(\kappa)$ can be described as a set $B(\kappa)$ of all sequences (m_1, m_2, \dots) of matrices $m_i \in M_{\kappa(i)}$ such that

$$\lim_{i \rightarrow \infty} \|m_{i+1} - m_i\| = 0$$

modulo the sequences converging to zero. The set $B(\kappa)$ is a C^* -algebra too, so $Q(\kappa)$ is a quotient C^* -algebra of $B(\kappa)$ modulo the ideal $I(\kappa)$ of sequences converging to zero. Moreover, the ideal $I(\kappa)$ is an essential ideal in $B(\kappa)$, so $B(\kappa)$ lies in the multiplier C^* -algebra $M(I(\kappa))$.

Note that though the C^* -algebras $Q(\kappa)$ consist only of α -invariant elements of $\prod_i M_{\kappa(i)} / \oplus_i M_{\kappa(i)}$, one should not think that $Q(\kappa)$ are smaller than the latter algebras. For example, consider $Q(\infty)$ and let $r = r(n)$ be a sequence of increasing integers with

$$\lim_{i \rightarrow \infty} (r(i+1) - r(i)) = \infty \quad (2)$$

then it is easy to see that the sequences (m_1, m_2, \dots) with $m_{r(i)} = 0$ form an ideal I_r of $Q(\infty)$ and the corresponding quotient C^* -algebra $Q(\infty)/I_r$ can be mapped to $\prod_i M_{\infty} / \oplus_i M_{\infty}$ by sending (m_1, m_2, \dots) to $(m_{r(1)}, m_{r(2)}, \dots)$. Using (2) one can lift any element of $\prod_i M_{\infty} / \oplus_i M_{\infty}$ by linear interpolation, hence this map is epimorphic.

Remark that the C^* -algebra $Q(\mathbf{1})$ is commutative, and its spectrum is the Higson corona, i.e. the quotient space of the Stone-Ćech corona of \mathbf{N} modulo the standard action of \mathbf{Z} . It is easy to see that the C^* -algebra $Q(\kappa)$ is unital if and only if $\kappa = \mathbf{n}$ for some finite n . In this case one has $Q(\mathbf{n}) = Q(\mathbf{1}) \otimes M_n$. Otherwise $Q(\kappa)$ contains the tensor product $Q(\mathbf{1}) \otimes M_{\infty}$ as a proper subalgebra. Indeed, the lifting of $Q(\mathbf{1}) \otimes M_{\infty}$ to $B(\kappa)$ consists of uniformly vanishing sequences $m = (m_1, m_2, \dots) \in B(\kappa)$. This means that for any such m and for any $\varepsilon > 0$ there exists a number j such that

$$\|(m_1, m_2, \dots) - (p_j m_1 p_j, p_j m_2 p_j, \dots)\| < \varepsilon,$$

where $p_j \in M_{\infty}$ denotes a projection onto the first j coordinates. For any κ approaching infinity it is easy to find an element of $B(\kappa)$, which is not uniformly vanishing. To do so consider a sequence $\{n_i\}$ with $\lim_{i \rightarrow \infty} (n_{i+1} - n_i) = \infty$ and a non-zero element $\lambda = (\lambda_1, \lambda_2, \dots) \in B(\mathbf{1})$ such that $\lambda_{n_i} = 0$ for all $i \in \mathbf{N}$. Let $\chi_{[n_i, n_{i+1}]}$ be the characteristic function for the set $[n_i, n_{i+1}] \subset \mathbf{N}$. Let $\{k_i\}$ be another sequence approaching infinity and

let $e_i \in M_\infty$ be the projection onto the i -th coordinate. Then the element $\oplus_i \chi_{[n_i, n_{i+1}]} \lambda e_{k_i}$ does not lie in $B(\mathbf{1}) \otimes M_\infty$.

2 Weak homotopic equivalence of $Q(\kappa)$

The C^* -algebras $Q(\kappa)$ are quotient algebras modulo an essential ideal, so it is natural to consider the corresponding weaker notion of homotopy for $*$ -homomorphisms into corona algebras. Remember that a function $g = g(t) : [0, 1] \rightarrow B(\kappa)$ is called continuous with respect to the strict topology [5] if the functions $g(t)k(t)$ and $k(t)g(t)$ are continuous in $C([0, 1], I(\kappa))$ for any continuous function $k = k(t) \in C([0, 1], I(\kappa))$. The C^* -algebra of all continuous functions with respect to the strict topology we denote by $C_{str}([0, 1], B(\kappa))$. The quotient maps $C_{str}([0, 1], B(\kappa)) \rightarrow C_{str}([0, 1], B(\kappa))/C([0, 1], I(\kappa))$ and $B(\kappa) \rightarrow Q(\kappa)$ we denote by q .

Let $f_0, f_1 : A \rightarrow Q(\kappa)$ be two $*$ -homomorphisms. They are called homotopic (as maps into quotient algebra) if there exists a family of maps (not necessarily $*$ -homomorphisms) $F_t : A \rightarrow B(\kappa)$ such that $F_t(a)$ is continuous with respect to the strict topology for any $a \in A$, the composition $q \circ F_t : A \rightarrow C([0, 1], Q(\kappa))$ is a $*$ -homomorphism and $q \circ F_i$ coincides with f_i , $i = 0, 1$.

As we are dealing with non-separable C^* -algebras, we have to consider even weaker homotopy equivalence. Let A, B be C^* -algebras. By $[A, B]$ we denote the set of all $*$ -homomorphisms from A to B .

Definition 2.1 We call A and B *weakly homotopy equivalent* if for any separable C^* -algebra C there are *natural* (with respect to C) maps

$$\phi = \phi_C : [C, A] \rightarrow [C, B] \quad \text{and} \quad \psi = \psi_C : [C, B] \rightarrow [C, A]$$

such that they preserve homotopy of $*$ -homomorphisms and that for any $f : C \rightarrow A$ and for any $g : C \rightarrow B$ the compositions $(\psi \circ \phi)(f)$ and $(\phi \circ \psi)(g)$ are homotopic to f and g respectively.

Lemma 2.2 *If A and B are weakly homotopy equivalent separable C^* -algebras then they are homotopy equivalent.*

Proof. Put $f = \phi_A(id_A) : A \rightarrow B$ and $g = \psi_B(id_B) : B \rightarrow A$. By f^* we denote the map $[B, X] \rightarrow [A, X]$ generated by the $*$ -homomorphism f . Then one has $f = f^*(id_B)$ and naturality of ψ implies that

$$\psi_A \phi_A(id_A) = \psi_A f^*(id_B) = f^* \psi_B(id_B) = f^* g = g \circ f$$

so by definition $g \circ f$ is homotopic to id_A . By the same way one can show that $f \circ g$ is homotopic to id_B . \square

Though this weak homotopy equivalence is very weak indeed, it preserves the K -theory.

Lemma 2.3 *If C^* -algebras A and B are weakly homotopy equivalent then $K_*(A) \cong K_*(B)$.*

Proof. It is easy to see that weak homotopy equivalence of C^* -algebras A and B implies the same equivalence of C^* -algebras $M_\infty \otimes A$ and $M_\infty \otimes B$. The groups K_0 and K_1 are defined by homotopy classes of $*$ -homomorphisms into these algebras of the separable C^* -algebras \mathbf{C} and $C_0(\mathbf{R})$ respectively. \square

The K -groups of the C^* -algebras $Q(\kappa)$ were calculated in [3].

Proposition 2.4 ([3]) *For any κ one has $K_0(Q(\kappa)) = \mathbf{Z}$ and $K_1(Q(\kappa)) = 0$. $K_0(Q(\kappa))$ is generated by the sequence (e_1, e_1, e_1, \dots) , where e_1 is the projection onto the first coordinate.* \square

Proposition 2.5 *Let κ_1 and κ_2 be two sequences approaching infinity but $\kappa_j \neq \infty$, $j = 1, 2$. Then the C^* -algebras $Q(\kappa_1)$ and $Q(\kappa_2)$ are homotopy equivalent.*

Proof. Let κ_3 be a sequence majorizing both κ_1 and κ_2 . Transitivity of homotopy equivalence shows that we can prove homotopy equivalence only for $Q(\kappa_1)$ and $Q(\kappa_3)$. As $\kappa_1(i) \leq \kappa_3(i)$ for any $i \in \mathbf{N}$, there exists a natural inclusion $f : Q(\kappa_1) \rightarrow Q(\kappa_3)$. To define a map g in the other direction consider an element $m = (m_1, m_2, \dots) \in B(\kappa_3)$, define numbers $r = r(i)$ by $\kappa_1(i-1) < \kappa_3(r) \leq \kappa_1(i)$ and put $(g(m))_i = m_r$. Then $g(m) \in B(\kappa_1)$ and $g(m) \in I(\kappa_1)$ whenever $m \in I(\kappa_3)$, so the map $g : Q(\kappa_3) \rightarrow Q(\kappa_1)$ is well-defined. Then the composition $g \circ f : Q(\kappa_1) \rightarrow Q(\kappa_1)$ is reduced to renumbering: a sequence $m = (m_1, m_2, \dots) \in B(\kappa_1)$ is mapped to a sequence

$$(\underbrace{m_1, \dots, m_1}_{s_1 \text{ times}}, \underbrace{m_2, \dots, m_2}_{s_2 \text{ times}}, \dots). \quad (3)$$

We are going to construct the homotopy $F_t : Q(\kappa_1) \rightarrow C_{str}([0, 1], B(\kappa_1))$ connecting $g \circ f$ with $id_{Q(\kappa_1)}$. For any $a \in Q(\kappa_1)$ choose a representative $m \in B(\kappa_1)$ of the form (3). For $t \in [0, 1/2]$ define $F_t(a)$ by connecting (by a linear path) the sequence (3) with the sequence

$$F_{1/2}(a) = (m_1, \underbrace{m_2, \dots, m_2}_{s_1+s_2-1 \text{ times}}, \underbrace{m_3, \dots, m_3}_{s_3 \text{ times}}, \dots). \quad (4)$$

For $t \in [1/2, 3/4]$ we connect the sequence (4) with the sequence

$$F_{3/4}(a) = (m_1, m_2, \underbrace{m_3, \dots, m_3}_{s_1+s_2+s_3-1 \text{ times}}, \underbrace{m_4, \dots, m_4}_{s_4 \text{ times}}, \dots),$$

etc. Finally we put $F_1(a) = (m_1, m_2, \dots)$. It is easy to see that $F_t(a)$ is continuous with respect to the strict topology (one has to check this only at $t = 1$), that $q \circ F_t(a)$ does not depend on choice of a representative m , and that $F_t(\cdot)$ is a $*$ -homomorphism modulo $C([0, 1], I(\kappa_1))$, so it gives the necessary homotopy. A similar homotopy connects $f \circ g$ with $id_{Q(\kappa_3)}$. \square

Proposition 2.6 *Let A be a separable C^* -algebra and let $f : A \rightarrow Q(\infty)$ be a $*$ -homomorphism. Then there exists a finite sequence κ such that $f(A) \subset Q(\kappa) \subset Q(\infty)$.*

Proof. Fix a dense sequence $\{a_i\}$, $i \in \mathbf{N}$, in A and a monotonely decreasing sequence of numbers $\{\varepsilon_n\}$, $n \in \mathbf{N}$, such that $\lim_{n \rightarrow \infty} \varepsilon_n = 0$. Let $m^{(i)} = (m_1^{(i)}, m_2^{(i)}, \dots) \in B(\infty)$ be liftings for $f(a_i)$. Then the numbers $\kappa(n)$ can be chosen to satisfy

$$\|m_j^{(i)} - p_{\kappa(n)} m_j^{(i)} p_{\kappa(n)}\| < \varepsilon_n \quad (5)$$

for all $1 \leq i, j \leq n$, where $p_{\kappa(n)} \in M_\infty$ are projections onto the first $\kappa(n)$ coordinates. It follows from (5) that for any i the sequences

$$(m_1^{(i)}, m_2^{(i)}, \dots) \quad \text{and} \quad (p_{\kappa(1)} m_1^{(i)} p_{\kappa(1)}, p_{\kappa(2)} m_2^{(i)} p_{\kappa(2)}, \dots)$$

give the same element in $Q(\infty)$ and the last sequence lies in $B(\kappa)$. As $Q(\kappa) \subset Q(\infty)$ is a C^* -subalgebra, f is continuous, and $f(a_i) \subset Q(\kappa)$, so $f(A) \in Q(\kappa)$ too. \square

Propositions 2.5 and 2.6 imply

Theorem 2.7 *If κ approaches infinity then all C^* -algebras $Q(\kappa)$ are weakly homotopy equivalent to each other.* \square

3 Growth functions and $Q(\kappa)$

Now we are going to show that the C^* -algebras $Q(\kappa)$ can be different for different growth of κ . We say that κ has polynomial growth if there exist numbers $0 < C_1 < C_2$ and $n \geq 1$ such that

$$C_1 i^n \leq \kappa(i) \leq C_2 i^n \quad \text{and} \quad \lim_i \frac{\kappa(i+1) - \kappa(i)}{i^n} = 0.$$

It has exponential growth if there exists a number $a > 1$ such that for all i one has $\kappa(i+1) \geq a \kappa(i)$.

Theorem 3.1 *Let κ_1 have polynomial growth and let κ_2 have exponential growth.*

1. *There exists a non-trivial $Q(1)$ -valued trace on $Q(\kappa_1)$.*

2. *There is no non-trivial trace on $Q(\kappa_2)$.*

Proof. 1. We define a trace on the C^* -algebra $B(\kappa_1)$ and then show that it vanishes on the ideal $I(\kappa_1)$. Let $m = (m_1, m_2, \dots) \in B(\kappa_1)$, $m_i \in M_{\kappa_1(i)}$. Put

$$\tau(m) = \left(\frac{\text{tr}_{\kappa_1(1)}(m_1)}{\kappa_1(1)}, \frac{\text{tr}_{\kappa_1(2)}(m_2)}{\kappa_1(2)}, \frac{\text{tr}_{\kappa_1(3)}(m_3)}{\kappa_1(3)}, \dots \right),$$

where tr_i is the standard trace on M_i normalized by $\text{tr}_i(1_i) = i$ for 1_i being the identity of M_i . If $\|m\| = 1$ then $\text{tr}_{\kappa_1(i)}(m_i) \leq \kappa_1(i)$, so $\|\tau(m)\| \leq 1$. One has also

$$\begin{aligned} \left\| \frac{\text{tr}_{\kappa_1(i+1)}(m_{i+1})}{\kappa_1(i+1)} - \frac{\text{tr}_{\kappa_1(i)}(m_i)}{\kappa_1(i)} \right\| &\leq \left\| \frac{\text{tr}_{\kappa_1(i+1)}(m_{i+1}) - \text{tr}_{\kappa_1(i)}(m_i)}{\kappa_1(i+1)} \right\| + \left\| \frac{\text{tr}_{\kappa_1(i)}(m_i)}{\kappa_1(i)\kappa_1(i+1)} \right\| \\ &\leq \|m_{i+1} - m_i\| + \frac{1}{\kappa_1(i+1)} \rightarrow 0, \end{aligned}$$

hence the image of τ lies in $B(\mathbf{1})$. Finally the property $\tau(ab) = \tau(ba)$ for $a, b \in B(\kappa_1)$ is obvious. So τ is a $B(\mathbf{1})$ -valued trace on $B(\kappa_1)$. It is also obvious that $\tau(I(\kappa_1)) \subset I(\mathbf{1})$, so τ is well-defined on the quotient algebra $Q(\kappa_1)$. It remains to show that τ is non-trivial. Let

$$m_i = \text{diag} \left\{ 1, \frac{\kappa_1(i) - 1}{\kappa_1(i)}, \dots, \frac{1}{\kappa_1(i)} \right\}$$

be diagonal matrices in $M_{\kappa_1(i)}$. Then

$$\|m_{i+1} - m_i\| = \frac{\kappa_1(i+1) - \kappa_1(i)}{\kappa_1(i+1)} \leq \frac{1}{C_1} \frac{\kappa_1(i+1) - \kappa_1(i)}{i^n} \rightarrow 0,$$

so the sequence $m = (m_1, m_2, \dots)$ lies in $B(\kappa_1)$, hence it defines an element in $Q(\kappa_1)$ and one has

$$\tau(m) = \left(\frac{\kappa_1(1) + 1}{2\kappa_1(1)}, \dots, \frac{\kappa_1(i) + 1}{2\kappa_1(i)}, \dots \right).$$

As $\lim_i \frac{\kappa_1(i)+1}{2\kappa_1(i)} = \frac{1}{2} \neq 0$, the trace τ is non-trivial.

2. For convenience of notation we assume that $a = 2$ and that $\kappa_2(i+1) \geq 2\kappa_2(i)$. We also assume that all M_i are embedded in the C^* -algebra of bounded operators on a Hilbert space H with a fixed basis ξ_1, ξ_2, \dots . Let u_1, u_2 be isometries on H such that $u_1(\xi_k) = \xi_{2k-1}$, $u_2(\xi_k) = \xi_{2k}$, $k \in \mathbf{N}$. Let $m = (m_1, m_2, \dots) \in B(\kappa_2)$. Then one can define an element $m \oplus m \in B(\kappa_2)$ as follows. Put $(m \oplus m)_1 = 0$ and

$$(m \oplus m)_{i+1} = u_1 m_i u_1^* + u_2 m_i u_2^*$$

for $i > 1$. Then $\dim(m \oplus m)_{i+1} = 2\kappa_2(i) \leq \kappa_2(i+1)$, and it is easy to see that $\|(m \oplus m)_{i+1} - (m \oplus m)_i\| \rightarrow 0$. As $\|(m \oplus m)_i\| \rightarrow 0$ whenever $m \in I(\kappa_2)$, so the map $m \mapsto (m \oplus m)$ defines a homomorphism $Q(\kappa_2) \rightarrow Q(\kappa_2)$. In a similar way one can define a direct sum of 4, 8, etc. summands.

Now suppose that there exists a trace τ on $Q(\kappa_2)$ with the property $\tau(ab) = \tau(ba)$ for $a, b \in Q(\kappa_2)$. Then τ extends to a trace on $B(\kappa_2)$ and we still denote this extension by τ . Let $m \in B(\kappa_2)$ be a positive element with $\|m\| = 1$ and $\tau(m) \neq 0$. One has $m \oplus m = m^{(1)} + m^{(2)}$, where

$$m^{(1)} = (0, u_1 m_1 u_1^*, u_1 m_2 u_1^*, u_1 m_3 u_1^*, \dots),$$

$$m^{(2)} = (0, u_2 m_1 u_2^*, u_2 m_2 u_2^*, u_2 m_3 u_2^*, \dots).$$

As all m_i are positive and of dimension $\kappa(i)$, so the sequences

$$a_j = (0, u_j(m_1)^{1/2}, u_j(m_2)^{1/2}, u_j(m_3)^{1/2}, \dots)$$

and

$$b_j = (0, (m_1)^{1/2}u_j^*, (m_2)^{1/2}u_j^*, (m_3)^{1/2}u_j^*, \dots),$$

$j = 1, 2$, are well-defined as elements of the algebra $B(\kappa_2)$. One has

$$\tau(m^{(j)}) = \tau(a_j b_j) = \tau(b_j a_j) = \tau(\alpha(m)),$$

where α is the shift (1), so $\tau(m^{(j)}) = \tau(m)$, as $m = \alpha(m)$ modulo $I(\kappa_2)$. Therefore $\tau(m \oplus m) = 2\tau(m)$. By the same way we can show that for any k one has

$$\underbrace{m \oplus m \oplus \dots \oplus m}_{2^k \text{ times}} \in B(\kappa_2)$$

and

$$\tau(\underbrace{m \oplus m \oplus \dots \oplus m}_{2^k \text{ times}}) = 2^k \tau(m). \quad (6)$$

On the other hand it is easy to see that

$$\|\underbrace{m \oplus m \oplus \dots \oplus m}_{2^k \text{ times}}\| = \|m\|,$$

hence

$$\tau(\underbrace{m \oplus m \oplus \dots \oplus m}_{2^k \text{ times}}) \leq 1 \quad (7)$$

and the contradiction between (6) and (7) shows that $\tau(m) = 0$ and finishes the proof. \square

Remember that a C^* -algebra A is called stable if $A \cong A \otimes M_\infty$.

Corollary 3.2 *If κ is of polynomial growth (or smaller) then the C^* -algebra $Q(\kappa)$ is not stable. If κ is of exponential growth then $Q(\kappa)$ is stable. \square*

Corollary 3.3 *Let κ be of polynomial growth. Let τ be a trace constructed in Theorem 3.1 and let $a \in Q(\mathbf{1}) \otimes M_\infty \subset Q(\kappa)$. Then $\tau(a) = 0$. \square*

Let κ be of polynomial growth. Consider a C^* -subalgebra $I_\tau \subset Q(\kappa)$ generated by all positive elements $a \in Q(\kappa)$ with $\tau(a) = 0$. As

$$\tau(ba) = \tau(a^{1/2}ba^{1/2}) \leq \|b\| \tau(a) = 0$$

for any $b \in Q(\kappa)$, so I_τ is an ideal in $Q(\kappa)$. By Lemmas 2.3 and 3.3 the K -groups of I_τ coincide with those of $Q(\kappa)$ and the inclusion $I_\tau \subset Q(\kappa)$ induces an isomorphism of their K -groups. Hence the six-term exact sequence shows that the quotient C^* -algebra $Q(\kappa)/I_\tau$ is K -contractible.

4 Continuous version for $Q(\kappa)$

Let $C_b([0, \infty), B)$ (resp. $C_0([0, \infty), B)$) be the C^* -algebra of bounded continuous (resp. vanishing at infinity) functions on the half-line taking values in a C^* -algebra B . The quotient algebra $Q_b(B) = C_b([0, \infty), B)/C_0([0, \infty), B)$ is usually used for describing continuous asymptotic homomorphisms. But this algebra is too big and one has often to make reparametrization in order to obtain slowly varying functions. Consider at first the case $B = M_\infty$. There exists a natural inclusion

$$Q(\infty) \longrightarrow Q_b(M_\infty), \quad (8)$$

given by the natural inclusion $\mathbf{N} \subset [0, \infty)$ and by linear interpolation from sequences of compact operators to the M_∞ -valued functions. This inclusion is obviously not epimorphic.

For a function $f(t) \in C_b([0, \infty), M_\infty)$ and for an interval $[a, b] \subset [0, \infty)$ put

$$\text{Var}_{a,b} f = \sup_{t_1, t_2 \in [a, b]} \|f(t_2) - f(t_1)\|$$

and consider in $C_b([0, \infty), M_\infty)$ a subset of functions with variation vanishing at infinity

$$C_b^{vv}([0, \infty), M_\infty) = \{f(t) \in C_b([0, \infty), M_\infty) : \lim_{t \rightarrow \infty} \text{Var}_{a+t, b+t} f = 0\}.$$

This subset is a C^* -subalgebra and it does not depend on choice of a and b . Denote by $Q_b^{vv}(M_\infty)$ the quotient C^* -algebra $C_b^{vv}([0, \infty), M_\infty)/C_0([0, \infty), M_\infty)$.

Lemma 4.1 *C^* -algebras $Q(\infty)$ and $Q_b^{vv}(M_\infty)$ are isomorphic.*

Proof. The image of $Q(\infty)$ under the map (8) obviously lies in $Q_b^{vv}(M_\infty)$. To get the map in the other direction one has to assign to a function $f(t) \in C_b^{vv}([0, \infty), M_\infty)$ representing an element of $Q_b^{vv}(M_\infty)$ a sequence $\{f(n)\}$ of its values at integer points. One can directly check that these $*$ -homomorphisms are inverse to each other. \square

Proposition 4.2 *The C^* -algebras $Q_b(M_\infty)$ and $Q_b^{vv}(M_\infty)$ are weakly homotopy equivalent.*

Proof. It is easy to see that for any $*$ -homomorphism of a separable C^* -algebra A into $Q_b(M_\infty)$ one can find a reparametrization $s = s(t)$ such that after passing from t to s the image of A lies in $Q_b^{vv}(M_\infty)$. \square

To imitate the continuous version for general κ one should consider telescopes. Let

$$T(\kappa) = \{f(t) \in C_b([0, \infty), M_\infty) : f(t) \in M_{\kappa(i)} \text{ for } t \in [i, i+1)\}$$

be the telescope C^* -algebra. Put

$$C_0([0, \infty), \kappa) = \{f(t) \in T(\kappa) : \lim_{t \rightarrow \infty} \|f(t)\| = 0\}$$

and

$$C_b^{vv}([0, \infty), \kappa) = \{f(t) \in T(\kappa) : \lim_{t \rightarrow \infty} \text{Var}_{t,t+1} f = 0\}.$$

Lemma 4.3 C^* -algebras $\mathcal{Q}(\kappa)$ and $C_b^{vv}([0, \infty), \kappa)/C_0([0, \infty), \kappa)$ are isomorphic. \square

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